

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. If you use any result that has not been proved in class you must provide a proof else you may not get enough marks.

(b) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

1. [12 Points] Let A, B, C be $n \times n$ matrices over a field F . If ABC is invertible, prove that B is invertible.

2. [36 Points] Let $v_1, \dots, v_n$ be linearly independent vectors in a vector space V over a field F . In each case below, determine the dimension of the subspace W spanned by the given vectors $w_1, \dots, w_n$.

- (i) $w_i = v_i - v_{i+1}$ for $i < n$ and $w_n = v_n - v_1$.
- (ii) $w_i = v_i + v_{i+1}$ for $i < n$ and $w_n = v_n + v_1$. (It may help to do the cases n is odd and n is even separately).
- (iii) $w_i = v_i - v_{n+1-i}$ for all i . (It may help to do the cases n is odd and n is even separately).

3. [26 Points] Define $U + W$ for subspaces U, W of a vector space V over a field F . Determine whether the following statements are True or False. If you claim a statement is True, you must provide a proof while if you claim it is False, you must provide a counterexample where it fails.

- (i) If U and W are finitely generated, then so is $U + W$.
- (ii) Every basis of $U + W$ contains a basis of U and a basis of W .
- (iii) If $S \subset U$ and $T \subset W$ are linearly independent subsets, then so is $S \cup T$.

4. [14 Points] Let W be a subspace of a vector space V over a field F . Prove the following statements about translates of W in V .

- (i) $W + v_1 = W + v_2$ iff $v_1 - v_2 \in W$.
- (ii) If $(W + v_1) \cap (W + v_2)$ is nonempty, then $W + v_1 = W + v_2$.

5. [12 Points] Define a linear map between two vector spaces over a field F . Give an example of an $\mathbb{R}$ -linear map $\mathbb{C} \rightarrow \mathbb{C}$ which is not $\mathbb{C}$ -linear. Justify your answer.